

Particle decay in inflationary cosmology

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We investigate the relaxation and decay of a particle during inflation by implementing the dynamical renormalization group. This investigation allows us to give a meaningful definition for the decay rate in an expanding universe. As a prelude to a more general scenario, the method is applied here to study the decay of a particle in de Sitter inflation via a trilinear coupling to massless conformally coupled particles, both for wavelengths much larger and much smaller than the Hubble radius. For superhorizon modes we find that the decay is of the form η^{Γ_1} with η being conformal time and we give an explicit expression for Γ_1 to leading order in the coupling which has a noteworthy interpretation in terms of the Hawking temperature of de Sitter space-time. We show that if the mass M of the decaying field is $\ll H$ then the decay rate during inflation is *enhanced* over the Minkowski space-time result by a factor $2H/\pi M$. For wavelengths much smaller than the Hubble radius we find that the decay law is $e^{-\frac{\alpha}{\kappa H} C(\eta)}$ with $C(\eta)$ the scale factor and α determined by the strength of the trilinear coupling. In all cases we find a substantial enhancement in the decay law as compared to Minkowski space-time. These results suggest potential implications for the spectrum of scalar density fluctuations as well as non-gaussianities.

I. INTRODUCTION

Inflation was originally proposed to solve several outstanding problems of the standard Big Bang model [1, 2, 3, 4, 5] thus becoming an important paradigm in cosmology. At the same time that inflation solves these problems it also provides a natural mechanism for the generation of scalar density fluctuations that seed large scale structure, thus explaining the origin of the temperature anisotropy in the cosmic microwave background (CMB), as well as tensor perturbations (primordial gravitational waves). Recently the Wilkinson Microwave Anisotropy Probe (WMAP) collaboration has provided a full-sky map of the temperature fluctuations of the cosmic microwave background (CMB) with unprecedented accuracy and an exhaustive analysis of the data confirming the basic and robust predictions of inflation [6, 7, 8].

During inflation quantum vacuum fluctuations are generated with physical wavelengths that grow faster than the Hubble radius, when the wavelength of these perturbations crosses the horizon these perturbations freeze out and decouple [2, 4, 5]. Wavelengths that are of cosmological relevance today re-enter the horizon during matter domination when the scalar (curvature) perturbations induce temperature anisotropies that are imprinted on the CMB at the last scattering surface [9, 10]. Generic inflationary models predict that these are mainly gaussian adiabatic perturbations with a spectrum that is almost scale invariant.

Inflationary dynamics is typically studied by treating the inflaton as a homogeneous classical scalar field [2, 3, 4] whose evolution is determined by a classical equation of motion, while quantum fluctuations of the scalar field around the classical value are treated in the gaussian approximation and provide the seeds for the scalar density perturbations of the metric. The quantum field theory interpretation of the classical homogeneous field configuration that drives inflation is that it is the expectation value of a quantum field operator in a translational invariant quantum state. There are important aspects of the dynamics that require a full quantum treatment for their consistent description, for example particle production and in particular particle *decay*. A systematic treatment of the quantum dynamics of the inflaton that includes particle production within a non-perturbative framework is given in ref. [11, 12].

While the dynamics of particle production during inflation has received much attention, the full quantum treatment of particle *decay* during the inflationary (or more generally during a rapidly expanding) stage has not been the focus of similar attention.

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While most reheating mechanisms rely on the coupling of the inflaton to other fields into which it can decay leading to a radiation dominated stage, the consequences of such coupling between the inflaton and other fields *during* the inflationary stage are typically neglected. In this article we focus on the study of the decay of a particle that could be the inflaton as a result of its coupling to other fields.

To the best of our knowledge a preliminary study of the decay of the inflaton during a de Sitter stage has been previously addressed within a particular case in ref.[13].

There could be several potentially important consequences of particle decay during inflation: if the inflaton couples to other particles, then its quantum fluctuations which seed density perturbations also couple to these other fields. Therefore the *decay* of the quantum fluctuations of the inflaton may result in a modification of the power spectrum of density perturbations. Furthermore the coupling of the quantum fluctuations of the inflaton and consequently of density perturbations to other fields may possibly induce non-gaussian correlations. For a recent review on non-gaussian correlations generated during inflation see ref.[14].

While a thorough assessment of these potentially relevant phenomena requires a detailed treatment of the coupling of gauge invariant perturbations to other fields into which these fluctuations can decay, in a spatially flat gauge there is a direct relation between the evolution equations for density perturbations and those of the quantum fluctuations of the inflaton[10].

Therefore if density perturbations also couple to these other fields as a consequence of the coupling of the inflaton field to these fields and *if* density perturbations decay into these fields, such decay implies that the amplitude of density perturbations will diminish with a consequent modification of their power spectrum.

Clearly a first step in the program to assess these potential observables is to understand the decay of fluctuations *during* inflation which is the focus of our study in this article.

These possibilities with distinct potential phenomenological consequences for CMB anisotropies motivate us to study in detail particle decay during an inflationary stage, which we take to be described by a de Sitter space-time.

The decay of the inflaton during a post-inflationary stage has been considered recently[15] as a possible source of metric (and therefore temperature) perturbations arising from an inhomogeneity of the inflaton coupling. However most of these treatments rely on the concept of the decay rate of a particle *in Minkowski space-time* seemingly uncritical of its validity in the (rapidly) expanding universe.

Goals of this article: In this article we study the decay of a particle into other particles during inflation. The decaying particle could be the inflaton but our study will be more generally valid. The main focus of our study is to provide an understanding of the concept of decay of a particle in a rapidly expanding cosmology, and to introduce and implement a method that allows a systematic and unambiguous study of the relaxational dynamics of quantum fields and in particular allows to extract the decay law resulting from interactions. In Minkowski space-time there are two alternative but equivalent manners to define the decay rate of a particle: I) the total decay rate is the inclusive transition probability *per unit time* from an initial 'in' state to final 'out' states, II) the total decay rate is the imaginary part of the space-time Fourier transform of the self-energy of the particle evaluated on the particle's mass shell and divided by its mass-shell energy. Both definitions are equivalent by dint of the optical theorem, or alternatively, unitarity. The calculation of a total decay rate from definition I) involves calculating the transition amplitude from some initial time $t_i \rightarrow -\infty$ to a final time $t_f \rightarrow +\infty$ and multiplying by its complex conjugate. In Minkowski space-time the transition amplitude from an asymptotic state in the past to an asymptotic state in the future is proportional to an energy conserving delta function. In squaring the amplitude, the square of this delta function is interpreted as the total time elapsed in the reaction (T) multiplying an energy conserving delta function. Dividing by the total time of the reaction (T) one extracts the decay rate. The calculation of the decay rate from the total width via definition II) requires that the self-energy be a function of the time difference and invokes energy-momentum conservation at each interaction vertex. The space-time Fourier transform of the self-energy features branch cut singularities in the complex frequency plane and the imaginary part across these cuts at the position of the particle mass shell gives the decay width or decay rate. The important point in this discussion is that in both cases the concept of a decay rate relies heavily on energy (and momentum) conservation. Herein lies the conceptual difficulty of extrapolating the concept of a decay rate (an inclusive transition probability per unit time) to the case of a rapidly expanding cosmology where there is no global timelike Killing vector associated with conservation of energy even when there may be space-like Killing vectors associated with spatial translational symmetries and momentum conservation. Such is the case for spatially flat Friedmann-Robertson-Walker cosmologies. The manifest lack of energy conservation in an expanding cosmology makes possible processes that would be forbidden in a static space-time by energy conservation[16]. In addition, contrary to Minkowski spacetime, cosmological modes in general do not decay exponentially with time, therefore the definition of the decay rate requires the kind of analysis we provide here.

The method:

Particle decay in de Sitter space-time has been previously studied in reference[13] for some very special cases that allowed a solution of the equation of motion. In this article we introduce a method that allows to study the relaxation of quantum fields and particle decay in great generality. The main strategy is to study the effective equations of

motion of the expectation values of fields as an initial value problem in linear response including the self-energy corrections. The solution of the equations of motion lead to an unambiguous identification of the decay law from the relaxation of the amplitude of the field as a consequence of the self-energy corrections (interactions). When self-energy corrections are included the equations of motion become non-local (non-Markovian) and cannot be solved in closed form in general.

When a perturbative solution of the equations of motion is attempted there emerge *secular terms*, namely terms that grow in time and invalidate the perturbative expansion. These secular terms indicate precisely the relaxation (or production) time scales. We implement the dynamical renormalization group introduced in [17] to provide a systematic resummation of these secular terms leading to the correct description of relaxation and decay. Such program has been successfully applied to a wide variety of non-equilibrium situations in Minkowski space time (see [17] and references therein).

In this article we generalize this approach to study particle decay during inflation. As a prelude to studying more general situations, we begin this program by implementing this method to study the decay of a massive and minimally coupled particle into conformally coupled massless scalars via a trilinear interaction vertex. After extracting the decay law to lowest order in the loop expansion for the self-energy, we study the limit of Minkowski space-time and show that the results obtained reproduce those familiar in Minkowski space time.

Brief summary of results: We introduce the dynamical renormalization group[17] method to study the relaxation of the expectation value of quantum fields as an initial value problem in the general case.

After introducing the method and discussing its systematic implementation in the general case we illustrate its application and study the decay of a massive particle (it could be the inflaton) coupled to conformally coupled massless particles via a trilinear vertex in de Sitter space time. This simpler setting allows to present the main aspects of the program as well as reveal the important features associated with the expansion in a clear manner. The relaxation and decay law is studied to lowest order in the coupling both for wavelengths that are inside and outside the Hubble radius during inflation. The decay constant for superhorizon modes have an interesting interpretation in terms of the Hawking temperature of de Sitter space-time. In all cases we find that the decay is *enhanced* during inflation as compared to the Minkowski space-time result. The decay law for modes deep within the horizon feature a wavevector dependence that leads to a larger suppression of the amplitude for longer wavelengths.

The article is organized as follows: in section II we introduce the model and the non-equilibrium Green's and correlation functions in arbitrary vacua, which are necessary ingredients for obtaining the effective equations of motion. In section III we obtain the equations of motion including self-energy corrections in the loop expansion and introduce the dynamical renormalization group method to extract the relaxation and decay law. In section IV we study specific cases up to leading order in the interaction and compare to the results in Minkowski space-time. In section V we summarize our results and conclusions and discuss potential implications of our results on the power spectrum of density fluctuations and non-gaussianity.

II. THE MODEL

We consider a spatially flat Friedmann-Robertson-Walker (FRW) cosmological space time with scale factor $a(t)$, in comoving coordinates the action is given by

$$A = \int d^3x dt a^3(t) \left\{ \frac{1}{2} \dot{\phi}^2 - \frac{(\nabla\phi)^2}{2a^2} - \frac{1}{2} (M^2 + \xi \mathcal{R}) \phi^2 + \frac{1}{2} \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{2a^2} - \frac{1}{2} (m^2 + \xi \mathcal{R}) \varphi^2 - g \phi \varphi^2 + J(t) \phi \right\} \quad (1)$$

with

$$\mathcal{R} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \quad (2)$$

being the Ricci scalar and ξ an arbitrary coupling to the Ricci scalar: $\xi = 0$ corresponds to minimal coupling and $\xi = 1/6$ corresponds to conformal coupling. The linear term in ϕ is a counterterm that will be used to cancel the tadpole diagram in the equations of motion.

It is convenient to pass to conformal time η with $d\eta = dt/a(t)$ and introduce a conformal rescaling of the fields

$$a(t) \phi(\vec{x}, t) = \chi(\vec{x}, \eta) \quad ; \quad a(t) \varphi(\vec{x}, t) = \delta(\vec{x}, \eta) . \quad (3)$$

The action becomes (after discarding surface terms that will not change the equations of motion)

$$A[\chi, \delta] = \frac{1}{2} \int d^3x d\eta \left\{ \frac{1}{2} \left[\chi'^2 - (\nabla\chi)^2 - \mathcal{M}_\chi^2(\eta) \chi^2 + \delta'^2 - (\nabla\delta)^2 - \mathcal{M}_\delta^2(\eta) \delta^2 \right] - gC(\eta) \chi \delta^2 - C^3(\eta) J(\eta) \chi \right\} , \quad (4)$$

with primes denoting derivatives with respect to conformal time η and

$$\mathcal{M}_\chi^2(\eta) = \left(M^2 + \xi_\chi \mathcal{R}\right) C^2(\eta) - \frac{C''(\eta)}{C(\eta)} \quad , \quad \mathcal{M}_\delta^2(\eta) = \left(m^2 + \xi_\delta \mathcal{R}\right) C^2(\eta) - \frac{C''(\eta)}{C(\eta)} \quad , \quad (5)$$

and $C(\eta) = a(t(\eta))$ is the scale factor as a function of conformal time. For inflationary cosmology the scale factor describes a de Sitter space-time, namely

$$a(t) = e^{Ht} \quad , \quad (6)$$

with H the Hubble constant and conformal time η is given by

$$\eta - \eta_0 = \frac{1}{H} (1 - e^{-Ht}) \quad , \quad (7)$$

where η_0 corresponds to the initial time $t = 0$. We choose

$$\eta_0 = -\frac{1}{H} \Rightarrow \eta = -\frac{e^{-Ht}}{H} \quad ; \quad C(\eta) = -\frac{1}{H\eta} \quad . \quad (8)$$

During inflation the effective time dependent masses of the fields are given by

$$\mathcal{M}_\chi^2(\eta) = \left[\frac{M^2}{H^2} + 12 \left(\xi_\chi - \frac{1}{6}\right)\right] \frac{1}{\eta^2} \quad , \quad \mathcal{M}_\delta^2(\eta) = \left[\frac{m^2}{H^2} + 12 \left(\xi_\delta - \frac{1}{6}\right)\right] \frac{1}{\eta^2} \quad . \quad (9)$$

The Heisenberg equations of motion for the spatial Fourier modes of wavevector k of the fields in the non-interacting ($g = 0$) theory are given by

$$\begin{aligned} \chi_k''(\eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu_\chi^2 - \frac{1}{4}\right)\right] \chi_k(\eta) &= 0 \\ \delta_k''(\eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu_\delta^2 - \frac{1}{4}\right)\right] \delta_k(\eta) &= 0 \quad , \end{aligned} \quad (10)$$

where

$$\nu_\chi^2 = \frac{9}{4} - \left(\frac{M^2}{H^2} + 12 \xi_\chi\right) \quad ; \quad \nu_\delta^2 = \frac{9}{4} - \left(\frac{m^2}{H^2} + 12 \xi_\delta\right) \quad . \quad (11)$$

The Heisenberg free field operators can be expanded in terms of the linearly independent solutions of the mode equation

$$S_\nu''(k; \eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu^2 - \frac{1}{4}\right)\right] S_\nu(k; \eta) = 0 \quad . \quad (12)$$

Two linearly independent solutions are given by

$$g_\nu(k; \eta) = \frac{1}{2} i^{-\nu-\frac{1}{2}} \sqrt{\pi\eta} H_\nu^{(2)}(k\eta) \quad (13)$$

$$f_\nu(k; \eta) = \frac{1}{2} i^{\nu+\frac{1}{2}} \sqrt{\pi\eta} H_\nu^{(1)}(k\eta) = [g_\nu(k; \eta)]^* \quad , \quad (14)$$

where $H_\nu^{(1,2)}(z)$ are Hankel functions. For wavevectors deep inside the Hubble radius $-k\eta \gg 1$ these functions have the asymptotic behavior

$$g_\nu(k; \eta) \stackrel{k\eta \rightarrow -\infty}{\approx} \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad , \quad f_\nu(k; \eta) \stackrel{k\eta \rightarrow -\infty}{\approx} \frac{1}{\sqrt{2k}} e^{ik\eta} \quad , \quad (15)$$

and are normalized so that their Wronskian is given by

$$W[g_\nu(k; \eta), f_\nu(k; \eta)] = g_\nu'(k; \eta) f_\nu(k; \eta) - g_\nu(k; \eta) f_\nu'(k; \eta) = -i \quad . \quad (16)$$

The most general solution of the mode equations (12) is given by

$$S_\nu(k; \eta) = C_1(k; \eta_0) f_\nu(k; \eta) + C_2(k; \eta_0) g_\nu(k; \eta) \quad , \quad (17)$$

where the coefficients $C_{1,2}$ are determined by an initial condition on the mode functions $S_\nu(k; \eta)$ at conformal time η_0 , namely

$$C_1(k; \eta_0) = -i[g_\nu(k; \eta_0) S'_\nu(k; \eta_0) - g'_\nu(k; \eta_0) S_\nu(k; \eta_0)] \quad (18)$$

$$C_2(k; \eta_0) = -i[f'_\nu(k; \eta_0) S_\nu(k; \eta_0) - f_\nu(k; \eta_0) S'_\nu(k; \eta_0)] . \quad (19)$$

The spatial Fourier transform of the free Heisenberg field operators $\chi_{\vec{k}}(\eta)$, $\delta_{\vec{k}}(\eta)$ are therefore written as

$$\chi_{\vec{k}}(\eta) = \alpha_{\vec{k}} S_{\nu_\chi}(k; \eta) + \alpha_{-\vec{k}}^\dagger S_{\nu_\chi}^*(k; \eta) \quad (20)$$

$$\delta_{\vec{k}}(\eta) = \beta_{\vec{k}} S_{\nu_\delta}(k; \eta) + \beta_{-\vec{k}}^\dagger S_{\nu_\delta}^*(k; \eta) , \quad (21)$$

where the Heisenberg operators $\alpha_{\vec{k}}$, $\alpha_{\vec{k}}^\dagger$ and $\beta_{\vec{k}}$, $\beta_{\vec{k}}^\dagger$ obey the usual canonical commutation relations.

Canonical commutation relations both for the Heisenberg fields (and their canonical momenta given by their derivatives with respect to conformal time) and the creation and annihilation operators entail that

$$|C_2(k; \eta_0)|^2 - |C_1(k; \eta_0)|^2 = 1 . \quad (22)$$

The vacuum state $|0\rangle$ is annihilated by $\alpha_{\vec{k}}$, $\beta_{\vec{k}}$. However a different choice of the coefficients $C_{1,2}$ determine a different choice of the vacuum state, the Bunch-Davies vacuum corresponds to choosing $C_2(k; \eta_0) = 1$, $C_1(k; \eta_0) = 0$ [18]. Heretofore we generically refer to the different choices of the coefficients $C_{1,2}$ constrained by Eq. (22), as S-vacua. An illuminating representation of these coefficients can be gleaned by computing the expectation value of the number operator in the Bunch-Davies vacuum. Consider the expansion of a scalar field either in terms of the Bunch-Davies basis $g_k(\eta)$ or in terms of the generalized basis $S(k; \eta)$, namely

$$\chi_{\vec{k}}(\eta) = a_{\vec{k}} g_{\nu_\chi}(k; \eta) + a_{-\vec{k}}^\dagger g_{\nu_\chi}^*(k; \eta) = \alpha_{\vec{k}} S_{\nu_\chi}(k; \eta) + \alpha_{-\vec{k}}^\dagger S_{\nu_\chi}^*(k; \eta) , \quad (23)$$

the creation and annihilation operators are related by a Bogoliubov transformation

$$\alpha_{\vec{k}}^\dagger = C_2 a_{\vec{k}}^\dagger - C_1 a_{-\vec{k}} , \quad \alpha_{\vec{k}} = C_2^* a_{\vec{k}} - C_1^* a_{-\vec{k}}^\dagger . \quad (24)$$

The Bunch-Davies vacuum $|0\rangle_{BD}$ is annihilated by $a_{\vec{k}}$, hence we find the expectation value

$${}_{BD}\langle 0 | \alpha_{\vec{k}}^\dagger \alpha_{\vec{k}} | 0 \rangle_{BD} = |C_1|^2 = N_k . \quad (25)$$

Where N_k is interpreted as the number of S-vacuum particles in the Bunch-Davies vacuum. In combination with the constraint (22) the above result suggests the following illuminating representation for the coefficients $C_{1,2}$

$$C_2(k) = \sqrt{1 + N_k} ; \quad C_1(k) = \sqrt{N_k} e^{i\theta_k} \quad (26)$$

where N_k and θ_k are real functions.

A. Non-equilibrium Green's and correlation functions

The main ingredients in the program to obtain the decay of the amplitude of the scalar field are the non-equilibrium Green's and correlation functions. Consider a generic free scalar field Φ quantized with the expansion

$$\Phi_{\vec{k}}(\eta) = \alpha_{\vec{k}} S_\nu(k; \eta) + \alpha_{-\vec{k}}^\dagger S_\nu^*(k; \eta) . \quad (27)$$

The non-equilibrium Green's and correlation functions are given by

$$G_k^{++}(\eta, \eta') = \langle 0 | T \left(\Phi_{\vec{k}}(\eta) \Phi_{-\vec{k}}(\eta') \right) | 0 \rangle = S_\nu(k; \eta) S_\nu^*(k; \eta') \Theta(\eta - \eta') + S_\nu^*(k; \eta) S_\nu(k; \eta') \Theta(\eta' - \eta) \quad (28)$$

$$G_k^{--}(\eta, \eta') = \langle 0 | \tilde{T} \left(\Phi_{\vec{k}}(\eta) \Phi_{-\vec{k}}(\eta') \right) | 0 \rangle = S_\nu(k; \eta) S_\nu^*(k; \eta') \Theta(\eta' - \eta) + S_\nu^*(k; \eta) S_\nu(k; \eta') \Theta(\eta - \eta') \quad (29)$$

$$G_k^{-+}(\eta, \eta') = \langle 0 | \Phi_{\vec{k}}(\eta) \Phi_{-\vec{k}}(\eta') | 0 \rangle = S_\nu(k; \eta) S_\nu^*(k; \eta') \quad (30)$$

$$G_k^{+-}(\eta, \eta') = \langle 0 | \Phi_{-\vec{k}}(\eta') \Phi_{\vec{k}}(\eta) | 0 \rangle = [G_k^{-+}(\eta, \eta')]^* = S_\nu^*(k; \eta) S_\nu(k; \eta') , \quad (31)$$

where T, \tilde{T} stand for the time and anti-time ordering symbols respectively. These Green's and correlation functions are *not* independent since they fulfill the following identity

$$G_k^{++}(\eta, \eta') + G_k^{--}(\eta, \eta') = G_k^{-+}(\eta, \eta') + G_k^{+-}(\eta, \eta') , \quad (32)$$

which can be trivially verified.

For generalized S-vacua, we find for example

$$G_k^{+-}(\eta, \eta') = [1 + N_k] g_\nu^*(k; \eta) g_\nu(k; \eta') + N_k g_\nu(k; \eta) g_\nu^*(k; \eta') + \sqrt{[1 + N_k]N_k} [e^{-i\theta_k} g_\nu(k; \eta) g_\nu(k; \eta') + c.c.] . \quad (33)$$

For physical wavelengths that are much smaller than the Hubble radius, namely for $k\eta, k\eta' \gg 1$ the two point correlation function above has the following behavior

$$G_k^{+-}(\eta, \eta') \stackrel{k\eta, k\eta' \rightarrow \infty}{\sim} \frac{1 + N_k}{2k} e^{-ik(\eta - \eta')} + \frac{N_k}{2k} e^{ik(\eta - \eta')} + \frac{1}{k} \sqrt{[1 + N_k]N_k} \cos[k(\eta + \eta') + \theta_k] . \quad (34)$$

The first two terms are similar to the Wightman function of a scalar field in a bath in equilibrium, whereas the last terms with the *sum* of the conformal times are a distinct feature of the mixing between particle and antiparticle states of the Bunch-Davies vacuum. We can impose asymptotically, for wavelengths deep within the Hubble radius that the physics be locally that of flat Minkowski space-time with a timelike Killing vector which implies that the two point function be translational invariant in time. This condition requires that the occupation numbers fulfill $N_k \rightarrow 0$ as $k \rightarrow \infty$. In particular if we further demand that the number of Bunch-Davies particles in a generalized S-vacuum be finite, then it must be that $N_k \lesssim \mathcal{O}(1/k^{3+\epsilon})$, $\epsilon > 0$ as $k \rightarrow \infty$.

III. EQUATIONS OF MOTION AND THE DYNAMICAL RENORMALIZATION GROUP

As mentioned in the introduction, in Minkowski space-time the decay rate of a particle can be obtained either from the imaginary part of the space-time Fourier transform of the retarded self-energy or alternatively from the transition probability per unit time. These methods are equivalent by dint of the optical theorem (unitarity) and both rely on energy conservation. In a rapidly expanding cosmology energy is not conserved (albeit there is covariant conservation) and the self-energy is not a function of the time difference but is a rapidly varying non-local function of the time arguments. In this situation a decay 'rate' cannot be identified unambiguously.

In ref.[17] we have introduced a method to study relaxation directly in *real time*. The method relies in obtaining the equation of motion of expectation values of the fields in linear response. The decay law (more general than a decay rate) is extracted from the time evolution of the *amplitude* of the expectation value. The equation of motion for the expectation value includes the non-local self-energy contributions in a consistent loop expansion. Although the equation is linear, it is generally non-local and when the self-energy is not time translational invariant (no energy conservation) it becomes an integro-differential equation which in general cannot be solved in closed form. The method introduced in ref.[17] relies on a *perturbative* expansion of the solution in terms of the coupling constant. However, such an expansion features secular terms, namely terms that grow in time and invalidate the perturbative expansion. The dynamical renormalization group[17] provides a consistent resummation of the series that leads to a uniform asymptotic expansion. The DRG improved solution directly allows to extract the decay law. This method has been applied and its applicability and reliability has been tested in a variety of equilibrium and non-equilibrium situations. The method is rather general and allows to resum secular terms for *any* set of differential or integro-differential evolution equations.

While the method has not yet been applied to the case of an expanding cosmology, we will confirm its reliability by analyzing the results in the limit when the expansion rate vanishes, namely, Minkowski space-time.

The method to study decay and relaxation begins by obtaining the equations of motion for the expectation value of the decaying field, which is obtained from the non-equilibrium generating functional[11, 12]

$$\mathcal{Z} = \int \mathcal{D}\chi^\pm \mathcal{D}\delta^\pm e^{i\{A[\chi^+, \delta^+] - A[\chi^-, \delta^-]\}} . \quad (35)$$

Where the superscripts \pm refer to the forward (+) and backward (−) branches along a closed contour in time corresponding to the unitary time evolution operator and its inverse, both of which are required to obtain expectation values in an initial density matrix (which could describe a pure state). For details on this method the reader is referred to ref.[11, 12].

Our goal is to obtain the equation of motion for the expectation value of the field χ . For this purpose, we implement the tadpole method by performing the following shift in the spatial Fourier transform of the field χ [11, 12]

$$\chi_k^\pm(\eta) = X_k^\pm(\eta) + \sigma_k^\pm(\eta) ; \quad \langle \chi_k^\pm(\eta) \rangle = X_k^\pm(\eta) ; \quad \langle \sigma_k^\pm(\eta) \rangle = 0 . \quad (36)$$

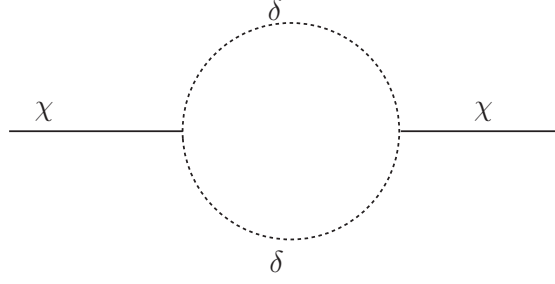


FIG. 1: Self energy loop of conformally coupled massless particles (dashed lines).

in the above expressions $\langle \dots \rangle$ stand for expectation values in the initial state which can be prepared by switching on an external source term to displace the field and switching the source off to let the field evolve. This is the usual method to prepare an initial value problem in linear response. For more details on this method we refer the reader to [11, 17].

In terms of the shifted field, the action for the spatial Fourier transformed fields becomes

$$A[X, \sigma^\pm, \delta^\pm] = \frac{1}{2} \int_{\eta_0}^0 d\eta \sum_{\vec{k}} \left\{ \sigma_{\vec{k}}'^\pm \sigma_{-\vec{k}}'^\pm - [k^2 + \mathcal{M}_\chi^2(\eta)] \sigma_{\vec{k}}^\pm \sigma_{-\vec{k}}^\pm + \delta_{\vec{k}}'^\pm \delta_{-\vec{k}}'^\pm - [k^2 + \mathcal{M}_\delta^2(\eta)] \delta_{\vec{k}}^\pm \delta_{-\vec{k}}^\pm \right\} \\ - \int_{\eta_0}^0 d\eta \sum_{\vec{k}} \left\{ \sigma_{-\vec{k}}^\pm \left[X_{\vec{k}}'' + (k^2 + \mathcal{M}_\chi^2(\eta)) X_{\vec{k}} \right] + g C(\eta) \frac{1}{\sqrt{V}} \sum_{\vec{q}} \left(\sigma_{\vec{k}}^\pm + X_{\vec{k}} \right) \delta_{\vec{q}} \delta_{-\vec{k}-\vec{q}} - C^3(\eta) J(\eta) X_{\vec{0}} \right\}$$

where we have neglected terms that cancel in the difference $A[X, \sigma^+, \delta^+] - A[X, \sigma^-, \delta^-]$, and V is the comoving spatial volume. The equation of motion for the expectation value $X_{\vec{k}}(\eta)$ is obtained by implementing the condition $\langle \sigma^\pm(\eta) \rangle = 0$ order by order in perturbation theory. The equation of motion is the same for the $+$ and $-$ conditions as a consequence of the identity (32), and up to one loop order it is given by

$$X_{\vec{k}}''(\eta) + [k^2 + \mathcal{M}_\chi^2(\eta)] X_{\vec{k}}(\eta) + 2g^2 C(\eta) \int_{\eta_0}^\eta d\eta' C(\eta') \mathcal{K}_k(\eta, \eta') X_{\vec{k}}(\eta') = 0, \quad (37)$$

where we have used the counterterm $J(\eta)$ to cancel the tadpole $\langle \delta_{\vec{q}} \delta_{-\vec{q}} \rangle$. At one loop order in the fields δ ($\mathcal{O}(g^2)$) the non-local kernel $\mathcal{K}_k(\eta, \eta')$ is given by the following expression

$$\mathcal{K}_k(\eta, \eta') = -i \int \frac{d^3 q}{(2\pi)^3} \left[G_{\delta, \vec{q}}^{-+}(\eta, \eta') G_{\delta, \vec{q}+\vec{k}}^{-+}(\eta, \eta') - G_{\delta, \vec{q}}^{+-}(\eta, \eta') G_{\delta, \vec{q}+\vec{k}}^{+-}(\eta, \eta') \right], \quad (38)$$

where the correlation functions $G_{\delta, \vec{q}}^{\pm\mp}(\eta, \eta')$ are those for the δ field given by the expressions Eqs.(28)-(31) with $\nu = \nu_\delta$ given in Eq. (11). This non-local kernel corresponds to the one-loop retarded self energy in real time as shown in fig. 1.

Although the equation of motion (37) is linear, it is non-local and a general solution is unavailable. However for weak coupling g^2 a perturbative solution can be found by writing

$$X_{\vec{k}}(\eta) = \sum_{n=0}^{\infty} (g^2)^n X_{n, \vec{k}}(\eta), \quad (39)$$

leading to the hierarchy of coupled equations

$$X_{0, \vec{k}}''(\eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu_\chi^2 - \frac{1}{4} \right) \right] X_{0, \vec{k}}(\eta) = 0, \quad (40)$$

$$X_{n, \vec{k}}''(\eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu_\chi^2 - \frac{1}{4} \right) \right] X_{n, \vec{k}}(\eta) = \mathcal{R}_n(k; \eta) \quad ; \quad n = 1, 2, \dots \quad (41)$$

$$\mathcal{R}_n(k; \eta) = -2 C(\eta) \int_{\eta_0}^\eta d\eta' C(\eta') \mathcal{K}_k(\eta, \eta') X_{n-1, \vec{k}}(\eta'). \quad (42)$$

In terms of two linearly independent solutions of the unperturbed equation $g_{\nu_\chi}(k, \eta); f_{\nu_\chi}(k, \eta)$ with Wronskian [see Eq. (16)] equal to $-i$, we write

$$X_{0,\vec{k}}(\eta) = A_{\vec{k}} g_{\nu_\chi}(k; \eta) + B_{\vec{k}} f_{\nu_\chi}(k; \eta) , \quad (43)$$

where the coefficients $A_{\vec{k}}, B_{\vec{k}}$ are determined by initial conditions. The hierarchy of coupled equations (40) can be solved iteratively by introducing the (retarded) Green's function of the second order differential operator on the left hand side of these equations. This Green's function is given by

$$\mathcal{G}(k; \eta, \eta') = i [g_{\nu_\chi}(k; \eta) f_{\nu_\chi}(k; \eta') - f_{\nu_\chi}(k; \eta) g_{\nu_\chi}(k; \eta')] \Theta(\eta - \eta') . \quad (44)$$

Hence the solution of the hierarchy of equations for $n \geq 1$ is given by

$$X_{n,\vec{k}}(\eta) = \int_{\eta_0}^0 d\eta' \mathcal{G}(k; \eta, \eta') \mathcal{R}_n(k; \eta') . \quad (45)$$

The general form of the perturbative solution Eq.(45) combined with the linearity of the equation of motion indicate that the full solution is of the form

$$X_{\vec{k}}(\eta) = A_{\vec{k}} g_{\nu_\chi}(k; \eta) [1 + g^2 F_1(k; \eta) + g^4 F_2(k; \eta) + \mathcal{O}(g^6)] + B_{\vec{k}} f_{\nu_\chi}(k; \eta) [1 + g^2 H_1(k; \eta) + g^4 H_2(k; \eta) + \mathcal{O}(g^6)] \quad (46)$$

where the functions $F_n(k; \eta); H_n(k; \eta)$ are found iteratively from the procedure described above. If the functions $F_n(k; \eta); H_n(k; \eta)$ remain bounded in the limit $\eta \rightarrow 0$ then the perturbative expansion provides a convergent uniform series. However, in general these functions feature *secular terms*, namely contributions that *diverge* in the limit $\eta \rightarrow 0$ and invalidate the perturbative expansion. In order to provide a uniform solution valid at all (conformal) times, these secular terms must be resummed via the dynamical renormalization group[17].

A. Dynamical renormalization group (DRG)

The general solution given by Eq. (46) highlights that the perturbative corrections can be interpreted as a *renormalization* of the complex amplitudes $A_{\vec{k}}; B_{\vec{k}}$, the dynamical renormalization group provides a systematic manner to resum the secular divergences in the perturbative expansion in terms of renormalization of the amplitudes.

This resummation program begins by extracting the secular terms of the functions $F_n(k; \eta), H_n(k; \eta)$ from the terms that remain finite and bounded in conformal time. Let us write

$$F_n(k; \eta) = F_{n,s}(k; \eta) + F_{n,f}(k; \eta) \quad ; \quad H_n(k; \eta) = H_{n,s}(k; \eta) + H_{n,f}(k; \eta) , \quad (47)$$

where $F_{n,s}(k; \eta), H_{n,s}(k; \eta)$ are secular, namely diverge in the limit $\eta \rightarrow 0$, whereas $F_{n,f}(k; \eta), H_{n,f}(k; \eta)$ remain bounded for $\eta \rightarrow 0$.

The dynamical renormalization group implements a resummation of the secular terms by introducing an arbitrary scale $\tilde{\eta}$ and a wave function renormalization of the complex amplitudes as follows[17]

$$A_{\vec{k}} = A_{\vec{k}}(\tilde{\eta}) Z_{\vec{k}}^A(\tilde{\eta}) \quad ; \quad Z_{\vec{k}}^A(\tilde{\eta}) = 1 + g^2 z_{1,\vec{k}}^A(\tilde{\eta}) + g^4 z_{2,\vec{k}}^A(\tilde{\eta}) + \mathcal{O}(g^6) , \quad (48)$$

$$B_{\vec{k}} = B_{\vec{k}}(\tilde{\eta}) Z_{\vec{k}}^B(\tilde{\eta}) \quad ; \quad Z_{\vec{k}}^B(\tilde{\eta}) = 1 + g^2 z_{1,\vec{k}}^B(\tilde{\eta}) + g^4 z_{2,\vec{k}}^B(\tilde{\eta}) + \mathcal{O}(g^6) . \quad (49)$$

The solution of the equation of motion now becomes

$$X_{\vec{k}}(\eta) = A_{\vec{k}}(\tilde{\eta}) g_{\nu_\chi}(k; \eta) [1 + g^2 (F_1(k; \eta) + z_{1,\vec{k}}^A(\tilde{\eta})) + \mathcal{O}(g^4)] + B_{\vec{k}}(\tilde{\eta}) f_{\nu_\chi}(k; \eta) [1 + g^2 (H_1(k; \eta) + z_{1,\vec{k}}^B(\tilde{\eta})) + \mathcal{O}(g^4)] \quad (50)$$

The coefficients $z_{n,\vec{k}}^{A,B}$ are chosen so that they cancel the secular terms $F_{n,s}(k; \eta), H_{n,s}(k; \eta)$ at the point $\eta = \tilde{\eta}$, namely

$$z_{1,\vec{k}}^A(\tilde{\eta}) = -F_{1,s}(k; \tilde{\eta}) \quad ; \quad z_{1,\vec{k}}^B(\tilde{\eta}) = -H_{1,s}(k; \tilde{\eta}) \quad ; \quad \text{etc.} \quad (51)$$

Therefore, the solution now becomes

$$X_{\vec{k}}(\eta) = A_{\vec{k}}(\tilde{\eta}) g_{\nu_\chi}(k; \eta) \{1 + g^2 [F_1(k; \eta) - F_{1,s}(k; \tilde{\eta})] + \mathcal{O}(g^4)\} + B_{\vec{k}}(\tilde{\eta}) f_{\nu_\chi}(k; \eta) \{1 + g^2 [H_1(k; \eta) - H_{1,s}(k; \tilde{\eta})] + \mathcal{O}(g^4)\} \quad (52)$$

This form of the solution can be written in the more illuminating manner

$$X_{\vec{k}}(\eta) = A_{\vec{k}}(\tilde{\eta}) g_{\nu_\chi}(k; \eta) \left\{ 1 + g^2 \int_{\tilde{\eta}}^{\eta} \frac{dF_{1,s}(k; \eta')}{d\eta'} d\eta' + \mathcal{O}(g^4) + \text{non-secular} \right\} + \\ + B_{\vec{k}}(\tilde{\eta}) f_{\nu_\chi}(k; \eta) \left\{ 1 + g^2 \int_{\tilde{\eta}}^{\eta} \frac{dH_{1,s}(k; \eta')}{d\eta'} d\eta' + \mathcal{O}(g^4) + \text{non-secular} \right\} \quad (53)$$

where the non-secular terms are terms **bounded** in the limit $\eta \rightarrow 0$. Eq.(53) reveals that the solution has been improved by choosing the scale $\tilde{\eta}$ arbitrarily close to η .

The solution $X_{\vec{k}}(\eta)$ is *independent* of the arbitrary renormalization scale $\tilde{\eta}$, namely

$$\frac{dX_{\vec{k}}(\eta)}{d\tilde{\eta}} = 0 \quad (54)$$

which leads to the *dynamical renormalization group equation*[17]. To lowest order the DRG equation is given by

$$\frac{\partial A_{\vec{k}}(\tilde{\eta})}{\partial \tilde{\eta}} - g^2 A_{\vec{k}}(\tilde{\eta}) \frac{\partial F_{1,s}(k; \tilde{\eta})}{\partial \tilde{\eta}} + \mathcal{O}(g^4) = 0, \quad (55)$$

$$\frac{\partial B_{\vec{k}}(\tilde{\eta})}{\partial \tilde{\eta}} - g^2 B_{\vec{k}}(\tilde{\eta}) \frac{\partial H_{1,s}(k; \tilde{\eta})}{\partial \tilde{\eta}} + \mathcal{O}(g^4) = 0, \quad (56)$$

To lowest order in g^2 the solution of these dynamical renormalization group equations is given by

$$A_{\vec{k}}(\tilde{\eta}) = A_{\vec{k}}(\tilde{\eta}_0) e^{g^2[F_{1,s}(k; \tilde{\eta}) - F_{1,s}(k; \tilde{\eta}_0)]} \quad (57)$$

$$B_{\vec{k}}(\tilde{\eta}) = B_{\vec{k}}(\tilde{\eta}_0) e^{g^2[H_{1,s}(k; \tilde{\eta}) - H_{1,s}(k; \tilde{\eta}_0)]}. \quad (58)$$

Since the scale $\tilde{\eta}$ is arbitrary, we can now set $\tilde{\eta} = \eta$ and obtain the renormalization group improved solution

$$X_{\vec{k}}(\eta) = A_{\vec{k}}(\eta) g_{\nu_\chi}(k; \eta) [1 + g^2 F_{1,f}(k, \eta) + \mathcal{O}(g^4)] + B_{\vec{k}}(\eta) f_{\nu_\chi}(k; \eta) [1 + g^2 H_{1,f}(k, \eta) + \mathcal{O}(g^4)], \quad (59)$$

$$A_{\vec{k}}(\eta) = A_{\vec{k}}(\eta_0) e^{g^2[F_{1,s}(k; \eta) - F_{1,s}(k; \eta_0)]}, \quad (60)$$

$$B_{\vec{k}}(\eta) = B_{\vec{k}}(\eta_0) e^{g^2[H_{1,s}(k; \eta) - H_{1,s}(k; \eta_0)]}, \quad (61)$$

the terms in the brackets in Eq. (59) are truly perturbatively small at all conformal times for weak coupling. That is, the dynamical renormalization group produces a perturbative expansion which is uniform in time. The exponential factors in the complex amplitudes (60)-(61) will determine the decay of the amplitude. The reliability and power of this method have been tested in many different cases and we refer the reader to[17] for a more thorough discussion.

We now implement this program in several relevant cases.

IV. SPECIFIC CASES

We begin our program by implementing the dynamical renormalization group resummation in a simpler case and to lowest order in the coupling, namely $\mathcal{O}(g^2)$, with the goal of highlighting the main aspects of the program in a simpler setting.

For this we consider the δ field (decay product) to be a massless conformally coupled field in its Bunch-Davies vacuum, namely $m = 0$, $\xi_\delta = 1/6$, $C_2 = 1$ and the decaying field χ to be massive and minimally coupled, namely $\xi_\chi = 0$. For the case of massless conformally coupled particles with Bunch-Davies vacuum in the self-energy loop

$$S_{\nu_\delta}(k; \eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad (62)$$

and the non-local kernel is given by

$$\mathcal{K}_k(\eta, \eta') = \int \frac{d^3 q}{(2\pi)^3} \frac{\sin[(q + |\vec{k} + \vec{q}|)(\eta - \eta')]}{2q|\vec{k} + \vec{q}|} = -\frac{1}{8\pi^2} \cos[k(\eta - \eta')] \mathcal{P}\left(\frac{1}{\eta - \eta'}\right) \quad (63)$$

where \mathcal{P} stands for the principal part. We *define* the principal part prescription as follows

$$\mathcal{P}\left(\frac{1}{\eta - \eta'}\right) \equiv \frac{\eta - \eta'}{(\eta - \eta')^2 + (\epsilon \eta')^2} = \frac{1}{2} \left[\frac{1}{\eta - \eta' + i\epsilon \eta'} + \frac{1}{\eta - \eta' - i\epsilon \eta'} \right] ; \quad \epsilon \rightarrow 0 \quad (64)$$

This prescription for the principal part regulates the short distance divergence in the operator product expansion with a dimensionless infinitesimal quantity ϵ independent of time. This η' -dependent point-splitting prescription in conformal time correspond to a time-independent point-splitting in comoving time $t = -\frac{1}{H} \log[-H \eta]$ [see Eq.(8)]:

$$\frac{\eta - \eta'}{(\eta - \eta')^2 + (\epsilon \eta')^2} = H e^{H t'} \frac{1 - e^{-H(t-t')}}{[1 - e^{-H(t-t')}]^2 + \epsilon^2}. \quad (65)$$

That is, a time splitting of ϵ/H between the points t and t' for $t \rightarrow t'$. This choice of regularization is consistent with the short-distance singularities of the operator product expansion in Minkowski space-time, and leads to a time-independent mass renormalization. Indeed, time-dependent mass renormalizations are allowed in cosmological space-times and they are associated with different regularization prescriptions. (For an analogous discussion using the moment cutoff instead of point splitting see sec. III of ref. [12]).

Repeating the calculations that follow but with a (conformal) time independent point-splitting ϵ instead of $\epsilon \eta'$ as conformal time separation in Eq.(64) leads to a (conformal) time dependent mass renormalization. While there is no unique choice of renormalization prescription, we demand a time independent renormalization of the mass, which is achieved by the principal part prescription adopted in Eq. (64).

Even with the simplification of conformally coupled massless fields in the self-energy loop, the study of the general case for arbitrary wavevectors is complicated by the fact that the solutions of the equations of motion at zero order are given by Eq. (43) with $g_{\nu_\chi}(k; \eta)$ given by Eq. (13) and ν_χ given by Eq. (11) for $\xi_\chi = 0$. However, progress can be made in the following relevant cases: i) $k\eta, k\eta' \ll 1$ corresponding to wavelengths that are larger than the Hubble radius all throughout inflation, which is equivalent to taking $k = 0$, ii) $k \neq 0$ with $|k\eta| \rightarrow 0$, this corresponds to modes that cross the horizon during inflation and iii) $k\eta, k\eta' \gg 1$ corresponding to wavelengths that are smaller than the Hubble radius all throughout inflation. We study each case separately.

A. Wavelengths larger than the Hubble radius: $k = 0$

In the case $k = 0$ it is convenient to choose the following linearly independent solutions of the unperturbed equations of motion

$$g_{\nu_\chi}(0, \eta) = (-\eta)^{\beta_+} \quad , \quad f_{\nu_\chi}(0, \eta) = (-\eta)^{\beta_-} \quad , \quad (66)$$

with

$$\beta_\pm = \frac{1}{2} \pm \nu_\chi \quad , \quad \nu_\chi = \sqrt{\frac{9}{4} - \frac{M^2}{H^2}} \quad , \quad (67)$$

in terms of which we write the solutions for the equation of motion at zeroth order as

$$X_{0,\vec{0}}(\eta) = a (-\eta)^{\beta_+} + b (-\eta)^{\beta_-} \quad , \quad (68)$$

where a and b are constant coefficients.

The retarded Green's function Eq.(44) necessary to solve the hierarchy of coupled equations, for $k = 0$ is given by

$$\mathcal{G}(\eta, \eta') = -\frac{1}{2\nu_\chi} [(-\eta)^{\beta_+} (-\eta')^{\beta_-} - (-\eta')^{\beta_+} (-\eta)^{\beta_-}] \Theta(\eta - \eta') = \frac{\sqrt{\eta \eta'}}{2\nu_\chi} \left[\left(\frac{\eta'}{\eta} \right)^{\nu_\chi} - \left(\frac{\eta}{\eta'} \right)^{\nu_\chi} \right] \Theta(\eta - \eta') \quad . \quad (69)$$

We start by computing $\mathcal{R}_1(\vec{0}, \eta)$ which from eqs. (42), (64) and (68) is given by

$$\mathcal{R}_1(\vec{0}, \eta) = -\frac{1}{8\pi^2 H^2 \eta} \int_{\eta_0}^{\eta} \frac{d\eta'}{\eta'} \frac{a (-\eta')^{\beta_+} + b (-\eta')^{\beta_-}}{\eta - \eta' + i\epsilon \eta'} + (\epsilon \rightarrow -\epsilon) \quad .$$

Expanding the kernel in a power series of the ratio η/η' and integrating term by term yields

$$\mathcal{R}_1(\vec{0}, \eta) = -\frac{1}{(2\pi H \eta)^2} \left\{ [a (-\eta)^{\beta_+} + b (-\eta)^{\beta_-}] (\log \epsilon + \gamma) + a \left[(-\eta)^{\beta_+} \psi(1 - \beta_+) + (-\eta_0)^{\beta_+} \sum_{k=1}^{\infty} \frac{1}{k - \beta_+} \left(\frac{\eta}{\eta_0} \right)^k \right] + \right.$$

$$+b \left[(-\eta)^{\beta_-} \psi(1 - \beta_-) + (-\eta_0)^{\beta_-} \sum_{k=1}^{\infty} \frac{1}{k - \beta_-} \left(\frac{\eta}{\eta_0} \right)^k \right] \Bigg\} , \quad (70)$$

where $\gamma = 0.57721 \dots$ is the Euler-Mascheroni constant and $\psi(z)$ stands for the logarithmic derivative of the Gamma function.

Inserting Eqs.(69) and (70) in Eq. (45) we find that the first order correction $X_{1,\vec{0}}(\eta)$ is given by

$$X_{1,\vec{0}}(\eta) = -\frac{1}{2(2\pi H)^2 \nu_\chi} \left(a (-\eta)^{\beta_+} \left\{ [\log \epsilon + \gamma + \psi(1 - \beta_+)] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right. \\ \left. - b (-\eta)^{\beta_-} \left\{ [\log \epsilon + \gamma + \psi(1 - \beta_-)] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right) , \quad (71)$$

where the non-secular terms are terms **bounded** in the limit $\eta \rightarrow 0$. Therefore, to first order in the coupling we find the solution of the equation of motion for the $\vec{k} = \vec{0}$ mode to be given by

$$X_{\vec{0}}(\eta) = a (-\eta)^{\beta_+} \left(1 - \frac{g^2}{2(2\pi H)^2 \nu_\chi} \left\{ [\log \epsilon + \gamma + \psi(\beta_-)] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right) + \\ + b (-\eta)^{\beta_-} \left(1 + \frac{g^2}{2(2\pi H)^2 \nu_\chi} \left\{ [\log \epsilon + \gamma + \psi(\beta_+)] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right) . \quad (72)$$

where we used that $\beta_+ + \beta_- = 1$ [Eq.(67)]. From this expression we can read off the secular contributions $F_{1,s}(0, \eta)$, $H_{1,s}(0, \eta)$ in Eq. (47) for $k = 0$, namely

$$F_{1,s}(0, \eta) = -\frac{1}{2(2\pi H)^2 \nu_\chi} [\log \epsilon + \gamma + \psi(\beta_-)] \ln \frac{\eta}{\eta_0} \quad (73)$$

$$H_{1,s}(0, \eta) = \frac{1}{2(2\pi H)^2 \nu_\chi} [\log \epsilon + \gamma + \psi(\beta_+)] \ln \frac{\eta}{\eta_0} \quad (74)$$

And the dynamical renormalization group resummation analyzed in section (III A) above leads to the following resummed solution

$$X_{\vec{0}}(\eta) = a(\tilde{\eta}_0) (-\eta)^{\beta_+} \left[\frac{\eta}{\tilde{\eta}_0} \right]^{-\frac{g^2 [\log \epsilon + \gamma + \psi(\beta_-)]}{2(2\pi H)^2 \nu_\chi}} [1 + \mathcal{O}(g^2)] + b(\tilde{\eta}_0) (-\eta)^{\beta_-} \left[\frac{\eta}{\tilde{\eta}_0} \right]^{\frac{g^2 [\log \epsilon + \gamma + \psi(\beta_+)]}{2(2\pi H)^2 \nu_\chi}} [1 + \mathcal{O}(g^2)] , \quad (75)$$

where the terms in the brackets are perturbatively small $[\mathcal{O}(g^2)]$ and have a finite limit as $\eta \rightarrow 0$. The above improved solution is uniform for all conformal time, however from this solution it is not clear what is the decay rate since the unperturbed solutions $(-\eta)^{\beta_\pm}$ are multiplied by *different* functions. In Minkowski space time the decay rate can be extracted straightforwardly and unambiguously because it describes in general an exponential relaxation of the amplitude that multiplies the oscillatory phases. However, in an expanding cosmology and in particular during a de Sitter stage, field modes with wavelengths larger than the Hubble radius do not propagate, they either grow or decay as a function of conformal (or comoving) time, thus the concept of the decay rate requires further examination.

As we explain in the following section, the decay of the amplitude can be separated from a mass renormalization in an unambiguous manner.

1. Decay rate and mass renormalization:

The relevant question that we must address is how to recognize a decay of the amplitude from a mass renormalization in this expression. We write the mass as the renormalized mass plus mass renormalization counterterms in the usual form

$$M^2 = M_R^2 + g^2 \delta M_1^2 + \mathcal{O}(g^4) . \quad (76)$$

Such renormalization results in a renormalization of ν_χ and of the exponents β_\pm , namely

$$\nu_\chi = \nu_{\chi,R} - \frac{g^2 \delta M_1^2}{2 \nu_{\chi,R} H^2} ; \quad \nu_{\chi,R} \equiv \sqrt{\frac{9}{4} - \frac{M_R^2}{H^2}} \quad (77)$$

$$\beta_\pm = \beta_{\pm,R} \mp \frac{g^2 \delta M_1^2}{2 \nu_{\chi,R} H^2} ; \quad \beta_{\pm,R} = \frac{1}{2} \pm \nu_{\chi,R} . \quad (78)$$

We now insert β_{\pm} as given Eq.(78) in Eq.(72) and to order g^2 we find

$$X_{\bar{0}}(\eta) = a (-\eta)^{\beta_+} \left(1 - \frac{g^2}{2 (2 \pi H)^2 \nu_{\chi}} \left\{ [\log \epsilon + \gamma + \psi(\beta_-) + (2 \pi)^2 \delta M_1^2] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right) + \\ + b (-\eta)^{\beta_-} \left(1 + \frac{g^2}{2 (2 \pi H)^2 \nu_{\chi}} \left\{ [\log \epsilon + \gamma + \psi(\beta_+) + (2 \pi)^2 \delta M_1^2] \log \frac{\eta}{\eta_0} + \text{non-secular} \right\} \right) . \quad (79)$$

In the expression above and in what follows we have suppressed the subscript R to avoid cluttering of notation, but it should be understood that all quantities are renormalized.

We see that mass renormalization cannot cancel both secular terms, and that the mass renormalization correction for the growing and decaying solutions have *opposite sign* . Therefore, choosing the mass renormalization counterterm δM_1^2 to be given by

$$\delta M_1^2 \equiv -\frac{1}{(2 \pi)^2} \left\{ \log \epsilon + \gamma + \frac{1}{2} [\psi(\beta_+) + \psi(\beta_-)] \right\} \quad (80)$$

cancels the logarithmic short distance divergence and leaves a finite contribution that multiplies *both* solutions equally.

With this choice of mass counterterm, the solution of the equation of motion to first order becomes

$$X_{\bar{0}}(\eta) = a (-\eta)^{\beta_+} \left[1 + \Gamma_1 \log \frac{\eta}{\eta_0} + \text{non-secular} \right] + b (-\eta)^{\beta_-} \left[1 + \Gamma_1 \log \frac{\eta}{\eta_0} + \text{non-secular} \right] . \quad (81)$$

where

$$\Gamma_1 = \frac{g^2 \tan \pi \nu_{\chi}}{16 \pi \nu_{\chi} H^2} . \quad (82)$$

and we used the relation

$$\psi \left(\frac{1}{2} + \nu \right) - \psi \left(\frac{1}{2} - \nu \right) = \pi \tan[\pi \nu]$$

We can now apply the DRG which exponentiates the secular terms and gives as the DRG-improved solution after mass renormalization

$$X_{\bar{0}}(\eta) = \left[\frac{\eta}{\tilde{\eta}_0} \right]^{\Gamma_1} \left\{ a(\tilde{\eta}_0) (-\eta)^{\beta_+} [1 + \mathcal{O}(g^2)] + b(\tilde{\eta}_0) (-\eta)^{\beta_-} [1 + \mathcal{O}(g^2)] \right\} . \quad (83)$$

We now clearly see that a decay rate Γ_1 can be unambiguously identified from the contribution that multiplies *both* solutions, whereas the mass renormalization enters with different signs for each solution.

Since the amplitudes $a(\tilde{\eta}_0), b(\tilde{\eta}_0)$ obey the dynamical renormalization group equations (55,56), the DRG improved solution (83) is *independent* of the scale $\tilde{\eta}_0$, a change in this scale is compensated by a change in the amplitudes.

The decay rate in de Sitter space-time can be read off from the above expression since by setting for simplicity $\tilde{\eta}_0 \equiv \eta_0$

$$\left[\frac{\eta}{\eta_0} \right]^{\Gamma_1} = e^{-\Gamma_1 H t} \equiv e^{-\Gamma_{dS} t} . \quad (84)$$

and $\Gamma_{dS} = H \Gamma_1$.

Eq.(82) can be analytically continued for $M > \frac{3}{2} H$ as

$$\Gamma_1 = \frac{g^2 \tanh \left[\pi \sqrt{\frac{M^2}{H^2} - \frac{9}{4}} \right]}{16 \pi H^2 \sqrt{\frac{M^2}{H^2} - \frac{9}{4}}} . \quad (85)$$

This result agrees with that obtained by a different method in ref.[13].

2. Minkowski space-time limit.

Minkowski space time is recovered in the limit $H \rightarrow 0$. In such limit we find from eqs. (67) and (85)

$$\begin{aligned} \nu_\chi &\stackrel{H \rightarrow 0}{=} i \frac{M}{H} \\ \Gamma_1 &\stackrel{H \rightarrow 0}{=} \frac{g^2}{16\pi M H} \tanh \left[\frac{\pi M}{H} \right] \stackrel{H \rightarrow 0}{=} \frac{g^2}{16\pi M H} \\ \beta_\pm &\stackrel{H \rightarrow 0}{=} \pm i \frac{M}{H} . \end{aligned} \quad (86)$$

Therefore, we find in this limit

$$\left[\frac{\eta}{\eta_0} \right]^{\Gamma_1} = e^{-\frac{g^2}{16\pi M} t} = e^{-\Gamma_M t} , \quad (87)$$

which displays the exponential decay of the amplitude with the correct decay rate $\Gamma_M = g^2/(16\pi M)$ for long-wavelength excitations with mass M in Minkowski space-time. Since $-\eta = e^{-Ht}/H$ up to an overall normalization of the field the dynamical renormalization group improved solution in this limit is given by

$$X_0(t) = e^{-\frac{g^2}{16\pi M} t} [A e^{iMt} + B e^{-iMt}] \quad (88)$$

which is the correct solution for a zero momentum excitation in Minkowski space-time.

Thus, clearly the dynamical renormalization group provides a consistent resummation program to study relaxation in a cosmological setting.

This comparison highlights that the decay rate in de Sitter space time is related to that of Minkowski space time as

$$\Gamma_{dS} \equiv \Gamma_1 H = \Gamma_M \frac{M}{\nu_\chi H} \tan[\pi \nu_\chi] . \quad (89)$$

For $M \ll H$ (for the inflaton this corresponds to the slow-roll limit) we find that the decay rate during inflation is *enhanced* as compared to that in Minkowski space-time

$$\frac{\Gamma_{dS}}{\Gamma_M} \stackrel{M \ll H}{=} \frac{2H}{\pi M} \gg 1 . \quad (90)$$

The decay rate $\Gamma_{dS} = H\Gamma_1$ with Γ_1 given by the result (85) has a noteworthy interpretation in terms of the Hawking temperature associated with the horizon in de Sitter space-time $T_H = \frac{H}{2\pi}$ [18] which can be seen as follows.

The solution of the free equation of motion for the zero mode given by Eq.(68) in comoving time is given by

$$X_0(t) = A e^{i\omega_+ t} + B e^{i\omega_- t} ; \quad \omega_\pm = \frac{3}{2} i \frac{H}{H} \pm \sqrt{M^2 - \frac{9}{4} H^2} . \quad (91)$$

The homogeneous modes are propagating for $M > 3H/2$. As compared to the case in Minkowski space-time we can identify ω^\pm as the complex poles that determine the free field evolution. In terms of these complex poles it is straightforward to see that for a propagating mode ($M > 3H/2$) Γ_{dS} can be written as follows.

$$\Gamma_{dS} = \frac{g^2}{16\pi} \frac{1 + 2N(\omega_+)}{\text{Re}(\omega_+)} , \quad (92)$$

with $N(\omega)$ the Bose-Einstein distribution function at the Hawking temperature, namely

$$N(\omega) = \frac{1}{e^{\frac{\omega}{T_H}} - 1} . \quad (93)$$

The expression (92) is similar to the expression for the decay rate in Minkowski space time at finite temperature and to lowest order in the coupling in terms of the pole frequency for the free field [19].

Thus at least in the case in which there is propagation, namely $M > 3H/2$, the decay rate can be identified as being that of Minkowski space time at the Hawking temperature. This noteworthy property of the decay rate for superhorizon modes has also been discussed in ref.[13].

B. Modes that cross the horizon during inflation $\vec{k} \neq 0$, $\eta \rightarrow 0^-$

For arbitrary k our integral equation (37) takes the form

$$X''_{\vec{k}}(\eta) + \left[k^2 - \frac{1}{\eta^2} \left(\nu_\chi^2 - \frac{1}{4} \right) \right] X_{\vec{k}}(\eta) - \left(\frac{g}{2\pi H} \right)^2 \frac{1}{\eta} \int_{\eta_0}^{\eta} \frac{d\eta'}{\eta'} \frac{(\eta - \eta') \cos k(\eta - \eta')}{(\eta - \eta')^2 + (\epsilon \eta')^2} X_{\vec{k}}(\eta') = 0 \quad (94)$$

where we used eqs.(63)-(64).

To first order in g^2 the solution $X_{1,\vec{k}}(\eta)$ given by Eq.(45) becomes

$$X_{1,\vec{k}}(\eta) = \frac{1}{(2\pi H)^2} \int_{\eta_0}^{\eta} \frac{d\eta'}{\eta'} \mathcal{G}(k; \eta, \eta') \int_{\eta_0}^{\eta'} \frac{d\eta''}{\eta''} \frac{(\eta' - \eta'') \cos k(\eta' - \eta'')}{(\eta' - \eta'')^2 + (\epsilon \eta'')^2} X_{0,\vec{k}}(\eta'') , \quad (95)$$

where $\mathcal{G}(k; \eta, \eta')$ is given by Eq.(44) and for simplicity we consider the solution

$$X_{0,\vec{k}}(\eta) = A_{\vec{k}} g_{\nu_\chi}(k; \eta) .$$

with $g_{\nu_\chi}(k; \eta)$ the mode function with Bunch-Davies initial condition. Eq.(95) can be written in the following form

$$X_{1,\vec{k}}(\eta) = A_{\vec{k}} \int_{\eta_0}^{\eta} \frac{d\eta'}{\eta'} g_{\nu_\chi}(k; \eta') J(\eta, \eta') , \quad (96)$$

where

$$J(\eta, \eta') = \frac{\sqrt{\eta}}{8\pi^2 H^2} \int_{\eta'}^{\eta} \frac{d\eta''}{\sqrt{\eta''}} \frac{(\eta' - \eta'') \cos k(\eta' - \eta'')}{(\eta' - \eta'')^2 + (\epsilon \eta'')^2} \text{Im} \left[H_{\nu_\chi}^{(1)}(k\eta) H_{\nu_\chi}^{(2)}(k\eta'') \right] . \quad (97)$$

Our goal is to evaluate $X_{1,\vec{k}}(\eta)$ for $\eta \rightarrow 0^-$. In order to achieve this we need $g_{\nu_\chi}(k; \eta)$ and the integrand in Eq.(97) for small arguments:

$$g_{\nu}(k; \eta) \stackrel{\eta \rightarrow 0^-}{=} \frac{\sqrt{\pi} \eta}{2} i^{-\nu-1/2} \left\{ \frac{i \Gamma(\nu)}{\pi} \left(\frac{2}{k \eta} \right)^\nu [1 + \mathcal{O}(k^2 \eta^2)] + \frac{1 - i \cot \pi \nu}{\Gamma(\nu + 1)} \left(\frac{k \eta}{2} \right)^\nu [1 + \mathcal{O}(k^2 \eta^2)] \right\} ,$$

$$\text{Im} \left[H_{\nu_\chi}^{(1)}(k\eta) H_{\nu_\chi}^{(2)}(k\eta') \right] \stackrel{\eta, \eta' \rightarrow 0^-}{=} \frac{1}{\pi \nu} \left[\left(\frac{\eta}{\eta'} \right)^\nu - \left(\frac{\eta'}{\eta} \right)^\nu \right] [1 + \mathcal{O}(\eta^2, \eta'^2)] . \quad (98)$$

Inserting Eq.(98) in Eq.(96) and (97) yields,

$$X_{1,\vec{k}}(\eta) \stackrel{\eta \rightarrow 0^-}{=} A_{\vec{k}} \frac{2^{\nu-2} \Gamma(\nu)}{\sqrt{\pi} \nu i^{\nu+3/2} k^\nu} \eta^{\frac{1}{2}-\nu} S \left(\frac{\eta_0}{\eta} \right) , \quad (99)$$

where

$$S \left(\frac{\eta_0}{\eta} \right) \stackrel{\eta \rightarrow 0^-}{=} -\frac{1}{2} \int_1^{\frac{\eta_0}{\eta}} \frac{dy}{y} (y^{-2\nu} - 1) \int_1^{\frac{\eta_0}{y\eta}} \frac{1}{(1+i\epsilon)t-1} \frac{dt}{t^{\nu+1/2}} + (\epsilon \rightarrow -\epsilon) . \quad (100)$$

Carrying out the integrations leads to the following result

$$\eta^{\frac{1}{2}-\nu} S \left(\frac{\eta_0}{\eta} \right) \stackrel{\eta \rightarrow 0^-}{=} \eta^{\frac{1}{2}-\nu} \left\{ \left[\psi \left(\nu + \frac{1}{2} \right) + \gamma + \ln \epsilon \right] \log \frac{\eta}{\eta_0} + \mathcal{O}(\eta_0) \right\} . \quad (101)$$

Notice that this $\eta \rightarrow 0^-$ behavior turns out to be k -independent. This is due to the fact that the term k^2 in Eq.(94) becomes negligible compared to the $\frac{1}{\eta^2}$ term for $\eta \rightarrow 0^-$ after the modes cross the horizon.

After mass renormalization according to Eq.(80) the logarithmic short distance singularity $\ln \epsilon$ is cancelled and we find for the mode functions the following result up to order g^2 ,

$$X_{0,\vec{k}}(\eta) + g^2 X_{1,\vec{k}}(\eta) \stackrel{\eta \rightarrow 0^-}{=} A_{\vec{k}} \frac{2^{\nu-1} \Gamma(\nu)}{\sqrt{\pi} i^{\nu-1/2} k^\nu} \eta^{\frac{1}{2}-\nu} \left[1 + \frac{g^2 \tan \pi \nu_\chi}{16 \pi \nu_\chi H^2} \log \frac{\eta}{\eta_0} + \mathcal{O}(\eta_0) \right] . \quad (102)$$

This result features the secular term $\log \eta$ which is resummed by implementing the DRG as in sec.IV-A with the result,

$$X_{\vec{k}}(\eta) \stackrel{\eta \rightarrow 0^-}{=} \frac{2^{\nu-1} \Gamma(\nu) \eta^{\frac{1}{2}-\nu}}{\sqrt{\pi} i^{\nu-\frac{1}{2}} k^\nu} A_{\vec{k}}(\tilde{\eta}_0) \left[\frac{\eta}{\tilde{\eta}_0} \right]^{\Gamma_1} [1 + \mathcal{O}(g^2)] , \quad (103)$$

where Γ_1 was defined in Eq.(82). It is clear from this result that the behaviour for $\eta \rightarrow 0^-$ and $k \neq 0$ with $|k\eta| \rightarrow 0$ is the same as that for the case $k = 0$ [see Eq.(83)]. This due to the fact that the physical wavenumbers $k\eta$ become so small for $\eta \rightarrow 0^-$ that they bear no relevance on the late time dynamics. While this result could be expected on physical grounds, it is important to see it emerge from the systematic implementation of the DRG method.

C. Wavelengths much smaller than the Hubble radius: $|k\eta| \gg 1$:

Anticipating mass renormalization we write the mass in the equation of motion (37) using Eq. (80). Furthermore for $|k\eta| \gg 1$ corresponding to modes with wavelengths much smaller than the Hubble radius during inflation, the hierarchy of equations of motion up to $\mathcal{O}(g^2)$ is given by

$$X''_{0,\vec{k}}(\eta) + k^2 X_{0,\vec{k}}(\eta) = 0 \quad (104)$$

$$X''_{1,\vec{k}}(\eta) + k^2 X_{1,\vec{k}}(\eta) = \mathcal{R}_1(k; \eta) , \quad (105)$$

with the inhomogeneity now given by

$$\mathcal{R}_1(k; \eta) = -\frac{\delta M_1^2}{H^2 \eta^2} X_{0,\vec{k}}(\eta) - 2 C(\eta) \int_{\eta_0}^{\eta} d\eta' C(\eta') \mathcal{K}_k(\eta, \eta') X_{0,\vec{k}}(\eta') . \quad (106)$$

where $\mathcal{K}_k(\eta, \eta')$ is given by Eq.(63).

The solution of the zeroth order equation is

$$X_{0,\vec{k}}(\eta) = A_k e^{-ik\eta} + B_k e^{ik\eta} . \quad (107)$$

The counterterm δM_1^2 is chosen to cancel the short distance divergence proportional to $\ln \epsilon / \eta^2$. After straightforward but lengthy algebra we find the following expression for the inhomogeneity in the limit $|k\eta_0| \gg |k\eta| \gg 1$

$$\mathcal{R}_1(k; \eta) = \frac{1}{8\pi^2 \eta^2 H^2} \left\{ A_k e^{-ik\eta} \left[\ln \frac{\eta}{\eta_0} + i \frac{\pi}{2} \right] + B_k e^{ik\eta} \left[\ln \frac{\eta}{\eta_0} - i \frac{\pi}{2} \right] + \dots \right\} , \quad (108)$$

where the dots stand for terms that are subleading in the limit $|k\eta| \gg 1$. The inhomogeneous equation for the first order correction can now be solved in terms of the retarded Green's function

$$\mathcal{G}(\eta, \eta') = \frac{1}{k} \sin[k(\eta - \eta')] \theta(\eta - \eta') . \quad (109)$$

To leading order in the limit $|k\eta_0| \gg |k\eta| \gg 1$ we find

$$X_{1,\vec{k}}(\eta) = -\frac{A_k e^{-ik\eta}}{32 \pi H k} \left\{ C(\eta) - C(\eta_0) - \frac{2i}{\pi H \eta} \ln \frac{\eta}{\eta_0} + \dots \right\} - \frac{B_k e^{ik\eta}}{32 \pi H k} \left\{ C(\eta) - C(\eta_0) + \frac{2i}{\pi H \eta} \ln \frac{\eta}{\eta_0} + \dots \right\} \quad (110)$$

where again the dots stand for terms that are subleading in the $|k\eta_0| \gg |k\eta| \gg 1$ limit and $C(\eta) = -1/H\eta$ is the scale factor. The term $C(\eta) - C(\eta_0)$ in the above expression is truly a *secular* term, since it grows by a factor larger than e^{60} during inflation. The validity of the perturbative expansion for this term is determined by the requirement that $|k/H C(\eta)| = |k\eta| \gg 1$, namely that the wavelengths are much smaller than the Hubble radius all throughout inflation.

Thus the solution up to this order is given by

$$\begin{aligned} X_{\vec{k}}(\eta) = & A_k e^{-ik\eta} \left\{ 1 - \frac{g^2}{32\pi H k} [C(\eta) - C(\eta_0)] + i \frac{g^2}{16\pi^2 H^2} \frac{\ln \frac{\eta}{\eta_0}}{k\eta} + \dots \right\} + \\ & + B_k e^{ik\eta} \left\{ 1 - \frac{g^2}{32\pi H k} [C(\eta) - C(\eta_0)] - i \frac{g^2}{16\pi^2 H^2} \frac{\ln \frac{\eta}{\eta_0}}{k\eta} + \dots \right\} , \end{aligned} \quad (111)$$

where the dots stand for terms that are of higher order in g^2 and subleading in the limit $|k\eta_0| \gg |k\eta| \gg 1$. The dynamical renormalization group resummation (59-61) leads to the following DRG improved solution

$$X_{\vec{k}}(\eta) = e^{-\frac{g^2}{32\pi H k} [C(\eta) - C(\eta_0)]} \left\{ A_k e^{-i[k\eta + \varphi_k(\eta)]} [1 + \mathcal{O}(g^4)] + B_k e^{i[k\eta + \varphi_k(\eta)]} [1 + \mathcal{O}(g^4)] \right\}, \quad (112)$$

where $\varphi_k(\eta)$ is a logarithmic phase that is not relevant for the decay of the amplitude, and the terms in the brackets are truly perturbative in the long time limit for wavelengths much smaller than the Hubble radius. In (112) we have chosen the renormalization scale $\tilde{\eta}_0$ to coincide with η_0 . The DRG improved solution (112) reveals the decay of the amplitude with the scale factor. The result above has the correct limit in Minkowski space-time as can be seen from the following argument. In comoving time, the difference $C(\eta) - C(\eta_0) = e^{Ht}$ therefore in the limit $H \rightarrow 0$

$$\frac{g^2}{32\pi H k} [C(\eta) - C(\eta_0)] \stackrel{H \rightarrow 0}{=} \frac{g^2}{32\pi k} t, \quad (113)$$

which gives the correct exponential relaxation of the amplitude of the field for large momentum in Minkowski space-time as shown in the appendix.

The results for the decay laws reproduce the decay rates in Minkowski space time in the limit $H \rightarrow 0$ [see Eqs. (87) and (113)] thus confirming the reliability of the dynamical renormalization group approach.

We can summarize the results obtained above as follows. Consider the solution $X_{0,\vec{k}}(\eta) = g_\nu(k; \eta)$ of the unperturbed equation with Bunch-Davies initial conditions as given in eq.(13). The asymptotic behavior of the power spectrum (here we do not include the k^3 normalization) of the unperturbed solution for modes deep inside the horizon $|k\eta| \gg 1$ and those well outside the horizon $|k\eta| \rightarrow 0$ is given by

$$\begin{aligned} |X_{0,\vec{k}}(\eta)|^2 &\stackrel{|k\eta| \gg 1}{=} \frac{1}{2k} \\ |X_{0,\vec{k}}(\eta)|^2 &\stackrel{|k\eta| \rightarrow 0}{=} \frac{2^{2\nu-2} \Gamma^2(\nu) \eta}{\pi (k\eta)^{2\nu}} \end{aligned} \quad (114)$$

Particle decay modifies the amplitude of the solution and consequently the power spectrum, which after the DRG resummation is given by

$$|X_{\vec{k}}(\eta)|^2 \stackrel{|k\eta| \gg 1}{=} \frac{1}{2k} e^{-\frac{g^2}{16\pi H k} [C(\eta) - C(\eta_0)]} \quad (115)$$

$$|X_{\vec{k}}(\eta)|^2 \stackrel{|k\eta| \rightarrow 0}{=} \frac{2^{2\nu-2} \Gamma^2(\nu) \eta}{\pi (k\eta)^{2\nu}} A_{\vec{k}}^2(\tilde{\eta}_0) \left[\frac{\eta}{\tilde{\eta}_0} \right]^{2\Gamma_1} \quad (116)$$

where we have normalized the mode functions to Bunch-Davies initial conditions at the beginning of inflation $\eta = \eta_0$ in Eq.(115). The solution for wavelengths larger than the Hubble radius is independent of the scale $\tilde{\eta}_0$ because the amplitude $A_{\vec{k}}(\tilde{\eta}_0)$ obeys the DRG equation Eq.(55). This amplitude at a given scale $\tilde{\eta}_0$ is obtained by matching the asymptotic forms of the DRG improved solution at a scale $\tilde{\eta}_0$. Clearly this amplitude will depend on the decay law of modes deep inside the horizon, which reflects a larger suppression of the amplitude for long wavelength modes.

These results are general hence they are also valid for the decay of the quantum fluctuations of the inflaton field. Since the quantum fluctuations of the inflaton field seed scalar density perturbations the result obtained above leads us to *conjecture* that the process of particle decay can lead to modifications of the power spectrum of superhorizon density perturbations which is obtained when the fluctuation freezes as $\eta \rightarrow 0^-$ [10]. The new renormalization scale $\tilde{\eta}_0$ will lead to violations of scale invariance much in the same way as in the renormalization group approach to deep inelastic scattering.

Clearly in order to assess the possibility of corrections to the power spectrum of density perturbations from decay of quantum fluctuations, the following issues must be studied further: i) a full gauge invariant treatment of the self-energy corrections to the equations of motion for density perturbation, ii) a DRG-improved solution for the whole range of momenta. Such program is necessarily beyond the scope of this article but the results above are suggestive of potentially important corrections to the power spectrum resulting from the decay of quantum fluctuations.

V. CONCLUSIONS AND DISCUSSION

The main goals of this article are a study of particle decay in inflationary cosmology, and to introduce and implement a method based on the dynamical renormalization group that allows to systematically obtain the relaxational dynamics of quantum fields and the decay law in particular.

One of the main points of this work is that during inflation or more generally during a period of very rapid cosmological expansion, the concept of a decay rate is ill suited to describe the relaxational dynamics or particle decay. In these cases of relevance in Early Universe cosmology, the lack of a global time-like Killing vector prevents the interpretation of a decay rate as an inclusive transition probability between asymptotic in and out states per unit time and deems unreliable the Minkowski space-time decay rate to describe particle decay.

The method that we propose to study the relaxational dynamics and to extract the decay law relies on the full quantum equation of motion of the expectation value of the field in an initial state in linear response. The quantum equations of motion are non-local as a consequence of loop corrections which determine the self-energy. The perturbative solution of these non-local equations of motion feature secular terms, namely terms that grow in time and lead to a breakdown of the naive perturbative expansion. The dynamical renormalization group[17] provides a systematic resummation of the perturbative expansion that leads to an unambiguous understanding of the decay law. The dynamical renormalization group program has been successfully implemented and tested in a variety of situations in Minkowski space-time (see [17]) and this work extends it to the case of expanding cosmology, in particular de Sitter space-time.

After introducing the method within a familiar model of interacting fields in inflationary cosmology we studied the relaxational dynamics of a massive field whose quanta decay into massless conformally coupled particles via a trilinear coupling. This model allows us to present the method and highlight several important aspects in a simpler setting. We have studied the relaxational dynamics and the decay law in the following cases: i) $k = 0$, namely superhorizon modes, ii) fixed $k \neq 0$ and $\eta \rightarrow 0^-$ ($|k\eta| \rightarrow 0$), namely modes that cross the horizon during inflation and iii) modes deep within the Hubble radius during inflation $|k\eta| \gg 1$.

Cases i) and ii) are found to be equivalent insofar as their relaxational dynamics. The decay constant in this case has a noteworthy interpretation in terms of the Hawking temperature of de Sitter space time and the Minkowski space-time limit $H \rightarrow 0$ reproduces the familiar decay rate of a massive particle into massless ones. In the case of modes that are deep inside the Hubble radius throughout inflation, we find that the relaxation is exponential in the *scale factor*, the amplitude decays as [see Eq. (112)] $e^{-\frac{g^2}{32\pi H k}[C(\eta)-C(\eta_0)]}$. We have confirmed that the limit $H \rightarrow 0$ reproduces the Minkowski space-time result.

In all cases studied here we find that the expansion *enhances* the decay. In the case of superhorizon modes we find that for $H \gg M$ (with M being the mass of the decaying field) the rate constant in de Sitter space time is larger than that in Minkowski space time by a factor $\sim H/M$.

Our results summarized in Eqs. (115)-(116) for the decay law of modes deep within the horizon as well as those that are superhorizon during inflation lead us to suggest potential observational implications. In an interacting theory the quantum fluctuations of the inflaton field will decay as a consequence of the interaction. These quantum fluctuations (in a suitable gauge [9, 10]) are the quantum seeds of metric perturbations. If these fluctuations decay as a consequence of the coupling between the inflaton and other fields (such a coupling is typically assumed for a post-inflationary reheating stage) then the decay of the amplitude both for modes deep within the horizon as well as those that cross the horizon during inflation will result in potential corrections to the power spectrum of density perturbations. In particular the amplitude of superhorizon modes will depend on the decay law of modes inside the horizon, which displays a larger suppression of the amplitude for modes of longer wavelength.

Furthermore, non-linear interactions are necessarily the source of non-gaussian correlation functions, in the case of a cubic vertex the three point function in the Born approximation also reveals the emergence of terms that grow in time [20]. This three point correlation function is a measure of non-gaussianity, and is clearly of interest to study further if and how the decay of inflaton fluctuations leads to non-gaussian correlations, perhaps by implementing the DRG as a resummation of the secular terms.

The decay of the inflaton field into other fields as well as the decay into *its own quanta* and the implications for the density fluctuations will be explored in a forthcoming article [21].

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APPENDIX A: DECAY RATE IN MINKOWSKI SPACE-TIME:

In order to compare the results obtained for the decay in inflationary cosmology with those more familiar in Minkowski space time we now summarize the Minkowski case. The equations of motion in Minkowski space time are

given by[17]

$$\ddot{X}_{\vec{k}}(t) + \omega_k^2 X_{\vec{k}}(t) + \int_{t_0}^t dt' K_k(t-t') X_{\vec{k}}(t') = 0 \quad ; \quad \omega_k^2 = k^2 + M^2 \quad (\text{A1})$$

where the non-local kernel is given by

$$K_k(t-t') = 2 g^2 \int \frac{d^3 q}{(2\pi)^3} \frac{\sin[(q + |\vec{k} + \vec{q}|)(t-t')]}{2q|\vec{k} + \vec{q}|} \quad (\text{A2})$$

This kernel can be written in terms of the spectral density $\sigma_k(\omega)$ in the form

$$\mathcal{K}_k(t-t') = -i \int_{-\infty}^{+\infty} d\omega \sigma_k(\omega) e^{-i\omega(t-t')} \quad (\text{A3})$$

with the spectral density given by

$$\sigma_k(\omega) = g^2 \int \frac{d^3 q}{16 \pi^3 q |\vec{k} + \vec{q}|} \left[\delta(\omega - q - |\vec{k} + \vec{q}|) - \delta(\omega + q + |\vec{k} + \vec{q}|) \right] \quad (\text{A4})$$

The delta functions represent the kinematic cuts for particles and antiparticles respectively. An analysis of the self-energy of the decaying particle reveals that the spectral function is related to the imaginary part of the retarded self-energy as follows[17]

$$\sigma_k(\omega) = \frac{1}{\pi} \text{Im}\Sigma(k; \omega) \quad (\text{A5})$$

where $\text{Im}\Sigma(k; \omega)$ is the imaginary part of the retarded self-energy. A straightforward calculation gives the following result

$$\sigma_k(\omega) = \frac{g^2}{8\pi^2} \text{sign}(\omega) \Theta(|\omega| - k) \quad (\text{A6})$$

The decay rate Γ_k is given by

$$\Gamma_k = \frac{\text{Im}\Sigma(k; \omega_k)}{2\omega_k} \quad (\text{A7})$$

where ω_k is the particle pole (mass shell dispersion relation) and leads to the result

$$\Gamma_k = \frac{g^2}{16 \pi \omega_k} \Theta(\omega_k - k) \quad (\text{A8})$$

The zero momentum limit gives

$$\Gamma_0 = \frac{g^2}{16\pi M} \quad (\text{A9})$$

which is the result quoted in Eq. (87) for the limit $H \rightarrow 0$.

The limit $M = 0$ which describes the Minkowski space-time limit of the case of wavelengths much smaller than the Hubble radius [see section IV C and Eq. (104)] yields,

$$\Gamma_k = \frac{g^2}{32\pi k} \quad (\text{A10})$$

since $\Theta(0) = 1/2$. This result coincides with the $H \rightarrow 0$ limit in Eq. (113). A detailed study using the dynamical renormalization group[17] reveals that if the particle mass shell coincides with the origin of the multiparticle threshold there emerges a phase that depends logarithmically on time. Such is the case in the massless limit and the logarithmic phase is the Minkowski space-time limit of the correction $\varphi_k(\eta)$ in the improved solution (112) (for more details see ref.[17]).

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